The Double Order Scaled Approximation (DOSA)

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This is a companion paper proving scaled integer edge weights can be used as in Thorup's algorithm for SSSP to retrieve shortest paths in two-dimensional obstacle maps in grid-space. The proof extends to n-dimensional spaces over the integers. Before getting started, we need to cover some ground covered in my paper [1] regarding digital curvature and pathfinding.

Definition 1 (Grid-space). *A space in which coordinates of points are in* $\mathbb{Z}^2$.

Definition 2 (Traversal graph). *A* **traversal graph** $G(V, E)$ *has vertices given by convex points along obstacle boundaries and edge weights given by* L_2 *distances between adjacent vertex pairs. A vertex pair is adjacent if the line between them does not intersect an obstacle boundary (see Figure 1).*

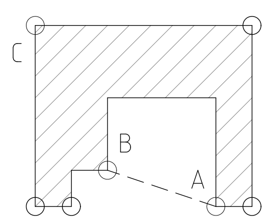

Figure 1: Vertices A and B are adjacent, vertices A and C are not.

We will say the vertex count along a path p is denoted by $|p|$, the i^{th} vertex, beginning with the initial vertex of p is denoted p_i, the distance between two vertices is denoted as the 2-normal distance between the points they represent (ie: $||p_{i+1} - p_i||_2$) and the length of p with real valued edge weights is given by the formula

$$||p||_2 = \sum_{i=1}^{|p|-1} ||p_{i+1} - p_i||_2 \tag{1}$$

If $u, v \in V$ are initial and final vertices of any path p and of any alternate path q such that

$$||q||_2 - ||p||_2 > \epsilon \tag{2}$$

where ϵ is the maximum error associated with a coordinate mapping, then a *good approximation of path length over the graph* G (denoted $\wr\wr x \wr\wr$) is such that $\wr\wr q \wr\wr > \wr\wr p \wr\wr$ for all such p and q. In our case, coordinates are mapped to $\mathbb{Z}^2$ from $\mathbb{R}^2$, so ϵ is 0.5 units.

In order to recover shortest paths with respect to $\mathbb{R}^2$ from a grid-space, we must find a good approximation. Our claim is that scaling our integer coordinates by about twice the order $(2|V|-1)$ of the traversal graph before taking the integer square root yields such an approximation. In our proof we will make use of the int function (3), which transforms a real number to its nearest integer, and this concept of the *Double Order Scaled Approximation* (DOSA).

$$\begin{aligned}
\mathrm{int}(x) &= \lfloor x + 0.5 \rfloor && (3)\\
\mathrm{int}^{(+)}(x) &= x + 0.5 \text{ for } x - \lfloor x \rfloor = 0.5 && (4)\\
\mathrm{int}^{(-)}(x) &\cong x - 0.5 \text{ for } x - \lfloor x \rfloor \lesssim 0.4999... && (5)
\end{aligned}$$

Definition 3 (Integer scaled approximation). *An approximation of an L_2 distance between points x_1 and x_2, each represented by a vertex in graph G is an* **integer scaled approximation** *if it is computed by the formula $||x_2 - x_1||_{\mathrm{int},G} = \mathrm{int}\left(||(sx_2) - (sx_1)||_2\right)$ where s is an arbitrary constant specific to G. Such an integer approximation is also said to be G-*integer scaled.

The following function is referred to as *the integer scaled approximation over G of the length of a path p*:

$$||p||_{\phi,G} = \sum_{i=1}^{|p|-1} ||p_{i+1} - p_i||_{\mathrm{int},G} \tag{6}$$

Theorem. *Suppose a grid-space obstacle map has a connected traversal graph $G(V,E)$. Then the integer scaled approximation is* good *for some $s \in \mathbb{Z}^+$.*

Proof. Let path p be any shortest path in graph G and let q be any alternate path between the initial and final vertices in p such that

$$||q||_2 - ||p||_2 > 0.5 \tag{7}$$

as in (2) with $\epsilon = 0.5$. Then the integer scaled approximation is good if and only if $||p||_{\phi,G} < ||q||_{\phi,G}$ for some $s \in \mathbb{Z}^+$. In the worst case, all edge weights in q will be rounded down by ϵ and all edge weights along p will be rounded up by ϵ.

Keep in mind we can assume the following: Because p and q have a final vertex u and an initial vertex v such that $u \neq v$, we know $|p| > 1$ and $|q| > 1$. The path p is a shortest path and any alternative q is longer than p by at least ϵ, which is 0.5. Since we need only show some $s > 0$ exists for the worst case, we may use $||q||_2 - ||p||_2 = 0.5$ from (7). Then the integer scaled approximation simplifies as follows:

$$||p||_{\phi,G} < ||q||_{\phi,G} \tag{8}$$

$$\sum_{i=1}^{|p|-1} \text{int}^{(+)}(||sp_{i+1} - sp_i||_2) < \sum_{i=1}^{|q|-1} \text{int}^{(-)}(||sq_{i+1} - sq_i||_2) \tag{9}$$

$$\left[\sum_{i=1}^{|p|-1} ||sp_{i+1} - sp_i||_2 + \epsilon\right] < \left[\sum_{i=1}^{|q|-1} ||sq_{i+1} - sq_i||_2 - \epsilon\right] \tag{10}$$

$$\left[\sum_{i=1}^{|p|-1} |s| \cdot ||p_{i+1} - p_i||_2 + \epsilon\right] < \left[\sum_{i=1}^{|q|-1} |s| \cdot ||q_{i+1} - q_i||_2 - \epsilon\right] \tag{11}$$

$$s \cdot \sum_{i=1}^{|p|-1} ||p_{i+1} - p_i||_2 + \sum_{i=1}^{|p|-1} \epsilon < s \cdot \sum_{i=1}^{|q|-1} ||q_{i+1} - q_i||_2 - \sum_{i=1}^{|q|-1} \epsilon \tag{12}$$

$$s \cdot ||p||_2 + \sum_{i=1}^{|p|-1} \epsilon < s \cdot ||q||_2 - \sum_{i=1}^{|q|-1} \epsilon \tag{13}$$

$$s \cdot ||p||_2 + (|p| - 1)(\epsilon) < s \cdot ||q||_2 - (|q| - 1)(\epsilon) \tag{14}$$

$$s \cdot ||p||_2 - s \cdot ||q||_2 < -(|p| - 1)(\epsilon) - (|q| - 1)(\epsilon) \tag{15}$$

$$(s)(||p||_2 - ||q||_2) < (-\epsilon)((|p| - 1) + (|q| - 1)) \tag{16}$$

$$s > -[(\epsilon)(|p| - 1 + |q| - 1)]/[(||p||_2 - ||q||_2)] \tag{17}$$

$$s > [(\epsilon)(|p| - 1 + |q| - 1)]/[(||q||_2 - ||p||_2)] \tag{18}$$

$$s > [(\epsilon)(|p| + |q| - 2)]/[(||q||_2 - ||p||_2)] \tag{19}$$

$$s > \frac{|p| + |q| - 2}{2(||q||_2 - ||p||_2)} \tag{20}$$

$$s > |p| + |q| - 2 \tag{21}$$

Thus for any shortest path p there exists an $s \in \mathbb{Z}^+$ such that the integer scaled approximation over the graph G is a good approximation of the path length of p. Since p is an arbitrary shortest path in G, p may be the weighted diameter of G – the longest shortest path in G – and so the integer scaled approximation with $s = |p| + |q| - 1$, satisfying (21) is a good approximation for all shortest paths in G. $\square$

Since finding the weighted diameter of a graph amounts to finding all pairs shortest paths, it would seem prudent to assume $|p|$ and $|q|$ are at most $|V|$, and thus letting $s = 2|V| - 1$ would suffice. From this formula we have the name Double Order Scaled Approximation.

References

[1] Jake Askeland and Winncy Du, *Fast exact convex and concave curvature in digital images with an application in path-finding*, Computational Geometry: Theory and Applications **Pending Publication** (2010).