

# The Double Order Scaled Approximation (DOSA), rev.2

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The purpose of this paper is to prove scaled integer edge weights can be used to retrieve shortest paths in two-dimensional obstacle maps in grid-space. The proof should extend to n-dimensional spaces over the integers. Before getting started, we need to cover some ground covered in my paper [1] regarding digital curvature and pathfinding.

**Definition 1 (Grid-space).** A space in which coordinates of points are in  $\mathbb{Z}^2$ .

**Definition 2 (Traversal graph).** A traversal graph (sometimes called a reachability graph)  $G(V, E)$  has vertices given by convex points along obstacle boundaries and edge weights given by  $L_2$  distances between adjacent vertex pairs. A vertex pair is adjacent if the line between them does not intersect an obstacle boundary (see Figure 1).

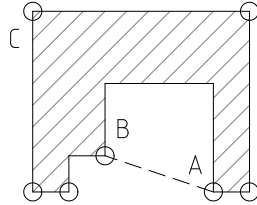


Figure 1: Vertices A and B are adjacent, vertices A and C are not.

We will say the vertex count along a path  $p$  is denoted by  $|p|$ , the  $i^{th}$  vertex, beginning with the initial vertex of  $p$  is denoted  $p_i$ , the distance between two vertices is denoted as the 2-normal distance between the points they represent (ie:  $\|p_{i+1} - p_i\|_2$ ) and the length of  $p$  with real valued edge weights is given by the formula

$$\|p\|_2 = \sum_{i=1}^{|p|-1} \|p_{i+1} - p_i\|_2 \quad (1)$$

If  $u, v \in V$  are initial and final vertices of any path  $p$  and of any alternate path  $q$  such that

$$\|q\|_2 - \|p\|_2 > \epsilon \quad (2)$$

where  $\epsilon$  is the maximum error associated with a coordinate mapping, then a *good approximation of path length over the graph  $G$*  (denoted  $\|x\|_G$ ) is such that  $\|q\|_G > \|p\|_G$  for all such  $p$  and  $q$ . In our case, coordinates are mapped to  $\mathbb{Z}^2$  from  $\mathbb{R}^2$ , so  $\epsilon$  is 0.5 units.

In order to recover shortest paths with respect to  $\mathbb{R}^2$  from a grid-space, we must find a good approximation. Our claim is that scaling our integer coordinates by about twice the order  $(2|V| - 1)$  of the traversal graph before taking the integer square root yields such an approximation. In our proof we will make use of the  $\text{int}$  function (3), which transforms a real number to its nearest integer, and this concept of the *Double Order Scaled Approximation* (DOSA). Equations (4) and (5) represent  $\text{int}$  when it has near maximal difference from the input, or maximal rounding error.

$$\text{int}(x) = \lfloor x + 0.5 \rfloor \quad (3)$$

$$\text{int}^{(+)}(x) = x + 0.5 \text{ for } x - \lfloor x \rfloor = 0.5 \quad (4)$$

$$\text{int}^{(-)}(x) \cong x - 0.5 \text{ for } x - \lfloor x \rfloor \lesssim 0.4999... \quad (5)$$

**Definition 3 (Integer scaled approximation).** An approximation of an  $L_2$  distance between points  $x_1$  and  $x_2$ , each represented by a vertex in graph  $G$  is an **integer scaled approximation** if it is computed by the formula  $\|x_2 - x_1\|_{\text{int},G} = \text{int}(\|(sx_2) - (sx_1)\|_2)$  where  $s$  is an arbitrary constant specific to  $G$ . Such an integer approximation is also said to be  $G$ -integer scaled.

The following function is referred to as *the integer scaled approximation over  $G$  of the length of a path  $p$* :

$$\|p\|_{\phi,G} = \sum_{i=1}^{|p|-1} \|p_{i+1} - p_i\|_{\text{int},G} \quad (6)$$

**Theorem.** Suppose a grid-space obstacle map has a connected traversal graph  $G(V, E)$ . Then the integer scaled approximation is good for some  $s \in \mathbb{Z}^+$ .

PROOF. Let path  $p$  be any shortest path in graph  $G$  and let  $q$  be any alternate path between the initial and final vertices in  $p$  such that

$$\|q\|_2 - \|p\|_2 > 0.5 \quad (7)$$

as in (2) with  $\epsilon = 0.5$ . Then the integer scaled approximation is good if and only if  $\|p\|_{\phi,G} < \|q\|_{\phi,G}$  for some  $s \in \mathbb{Z}^+$ . In the worst case, all edge weights in  $q$  will be rounded down by  $\epsilon$  and all edge weights along  $p$  will be rounded up by  $\epsilon$ . In other words, when calculating shortest paths, path  $p$  will have a total path-length error due to rounding of  $L_2$  distances of  $\|p\|_{\phi,G} - \|p\| = |p|\epsilon > 0$ . Similarly,  $q$  will have a total path-length error of  $\|q\| - \|q\|_{\phi,G} = |q|\epsilon < 0$ .

Keep in mind we can assume the following: Because  $p$  and  $q$  have a final vertex  $v$  and an initial vertex  $u$  and  $u \neq v$  so we know  $|p| > 1$  and  $|q| > 1$ . The path  $p$  is a shortest path and any alternative  $q$  is longer than  $p$  by at least  $\epsilon$ , which is 0.5. Since we need only show some  $s > 0$  exists for the worst case (where the paths are very close to the same length), we may use  $\|q\|_2 - \|p\|_2 = 0.5$  from (7). Then the integer scaled approximation simplifies as follows:

$$\|p\|_{\phi,G} < \|q\|_{\phi,G} \quad (8)$$

$$\sum_{i=1}^{|p|-1} \text{int}^{(+)}(\|sp_{i+1} - sp_i\|_2) < \sum_{i=1}^{|q|-1} \text{int}^{(-)}(\|sq_{i+1} - sq_i\|_2) \quad (9)$$

$$\left[ \sum_{i=1}^{|p|-1} \|sp_{i+1} - sp_i\|_2 + \epsilon \right] < \left[ \sum_{i=1}^{|q|-1} \|sq_{i+1} - sq_i\|_2 - \epsilon \right] \quad (10)$$

$$\left[ \sum_{i=1}^{|p|-1} |s| \cdot \|p_{i+1} - p_i\|_2 + \epsilon \right] < \left[ \sum_{i=1}^{|q|-1} |s| \cdot \|q_{i+1} - q_i\|_2 - \epsilon \right] \quad (11)$$

$$s \cdot \sum_{i=1}^{|p|-1} \|p_{i+1} - p_i\|_2 + \sum_{i=1}^{|p|-1} \epsilon < s \cdot \sum_{i=1}^{|q|-1} \|q_{i+1} - q_i\|_2 - \sum_{i=1}^{|q|-1} \epsilon \quad (12)$$

$$s \cdot \|p\|_2 + \sum_{i=1}^{|p|-1} \epsilon < s \cdot \|q\|_2 - \sum_{i=1}^{|q|-1} \epsilon \quad (13)$$

$$s \cdot \|p\|_2 + (|p| - 1)(\epsilon) < s \cdot \|q\|_2 - (|q| - 1)(\epsilon) \quad (14)$$

$$s \cdot \|p\|_2 - s \cdot \|q\|_2 < -(|p| - 1)(\epsilon) - (|q| - 1)(\epsilon) \quad (15)$$

$$(s)(\|p\|_2 - \|q\|_2) < (-\epsilon)(|p| + |q| - 2) \quad (16)$$

$$(s)(-1/2) < (-\epsilon)(|p| + |q| - 2) \quad (17)$$

$$s > (-\epsilon)(|p| + |q| - 2)/(-1/2) \quad (18)$$

$$s > (2\epsilon)(|p| + |q| - 2) \quad (19)$$

$$s > |p| + |q| - 2 \quad (20)$$

Thus for any shortest path  $p$  there exists an  $s \in \mathbb{Z}^+$  such that the integer scaled approximation over the graph  $G$  is a good approximation of the path length of  $p$ . Since  $p$  is an arbitrary shortest path in  $G$ ,  $p$  may be the weighted diameter of  $G$  – the longest shortest path in  $G$  – and so the integer scaled approximation with  $s = |p| + |q| - 1$ , satisfying (20) is a good approximation for all shortest paths in  $G$ .  $\square$

Since the work of finding the weighted diameter of a graph amounts to the work of finding all pairs shortest paths, one can just assume  $|p|$  and  $|q|$  are at most  $|V|$ , given  $s = 2|V| - 1$ . From this formula we have the name Double Order Scaled Approximation.

## References

- [1] Jake Askeland and Winncy Du, *Fast exact convex and concave curvature in digital images with an application in path-finding*, **Working Paper** (2010).